Муавр теоремасы

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Комплекс санның тригонометриялық түрі

Тригонометриялық түрдегі комплекс сандарды көбейту

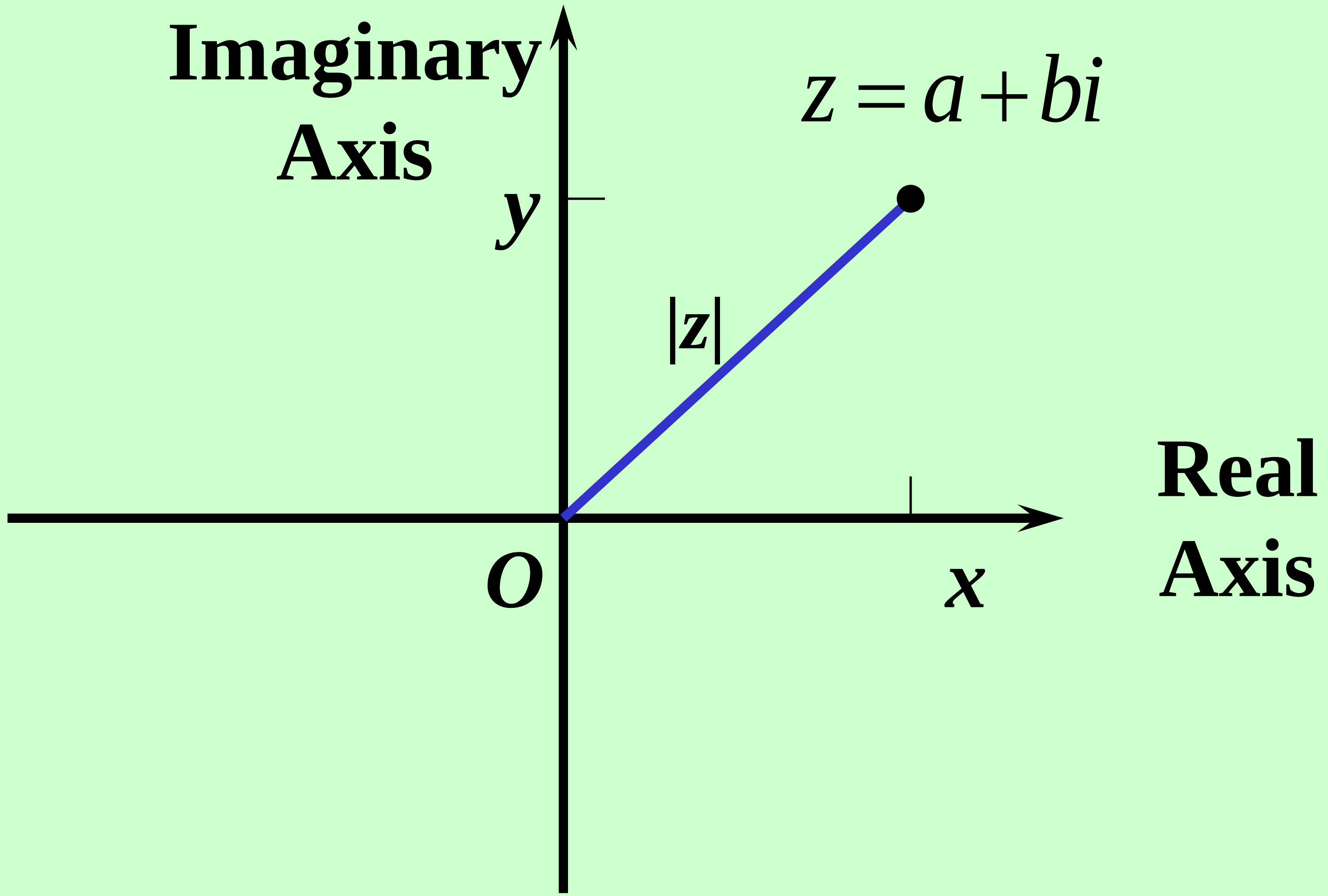
Тригонометриялық түрдегі комплекс сандарды бөлу

Тригонометриялық түрдегі комплекс сандарды дәрежеге Шығару (Муавр формуласы)

Комплекс сандардың түбірлері

Комплекс сандар

- $z = x + yi$ комплекс санын геометриялық тұрғыдан комплекс жазықтықтағы (х, у)нүктесі деп қарауға болады. х-осі нақты ось, у-осі жорамал ось деп аталады.


Imaginary Axis
z = a + bi
y
|z|
Real Axis
O
x

Z комплекс санының модулі

- z = x + yi комплекс саны берілсін. z комплекс санының |z| модулі деп (x, y) нүктесінің бас нүктеден арақашықтығын айтады.

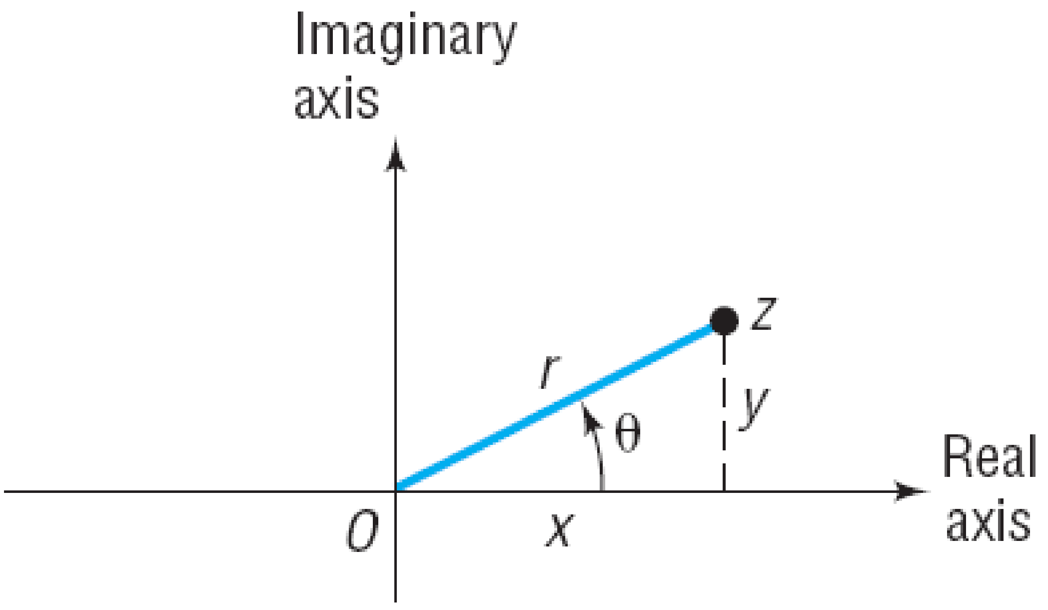
$$|z| = \sqrt{x^2 + y^2}$$

Комплекс санның тригонометриялық формасы

- Егер $0 \leq \theta \leq 2\pi$ болса, онда $z = x + yi$ комплекс санын тригонометриялық формада $z = x + yi = (r \cos \theta) + (r \sin \theta)i$ немесе $z = r (\cos \theta + i \sin \theta)$ жазуға болады
- θ бұрышы z комплекс санның аргументі
- $|z| = r$

If $r \geq 0$ and $0 \leq \theta < 2\pi$, the complex number $z = x + yi$ may be written in **polar form** as

$$z = x + yi = (r\cos\theta) + (r\sin\theta)i = r(\cos\theta + i\sin\theta) \qquad \textbf{(4)}$$



$z = x + yi = r(\cos\theta + i\sin\theta)$,
$r \geq 0,\ 0 \leq \theta < 2\pi$

$$|z| = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}$$

$$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} =$$

$$= \sqrt{r^2(\cos^2\theta + \sin^2\theta)}$$

$$= \sqrt{r^2} = r$$

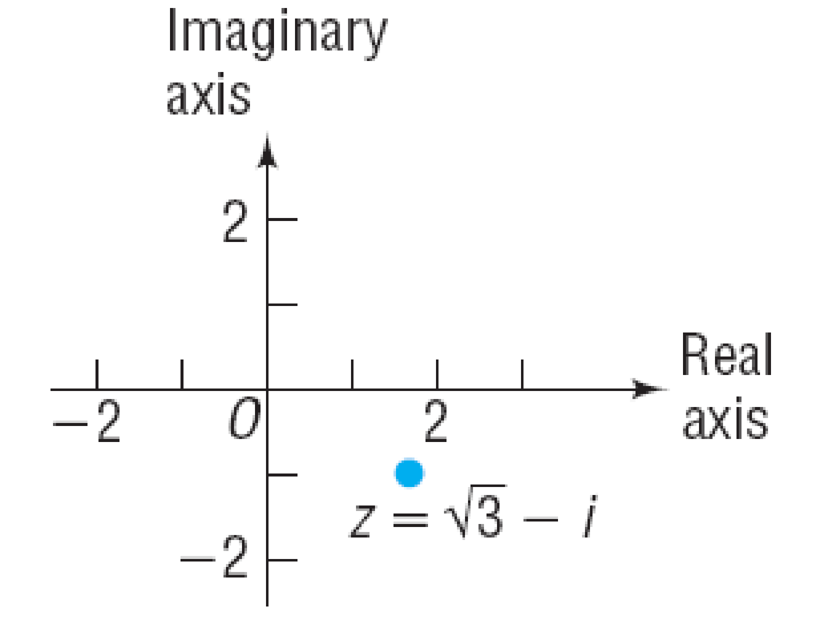
$$|z| = r$$

$$r(\cos\theta + i\sin\theta)$$

Тригонометриялық формaны қысқаша жазуға болады

$$r(cis\,\theta)$$

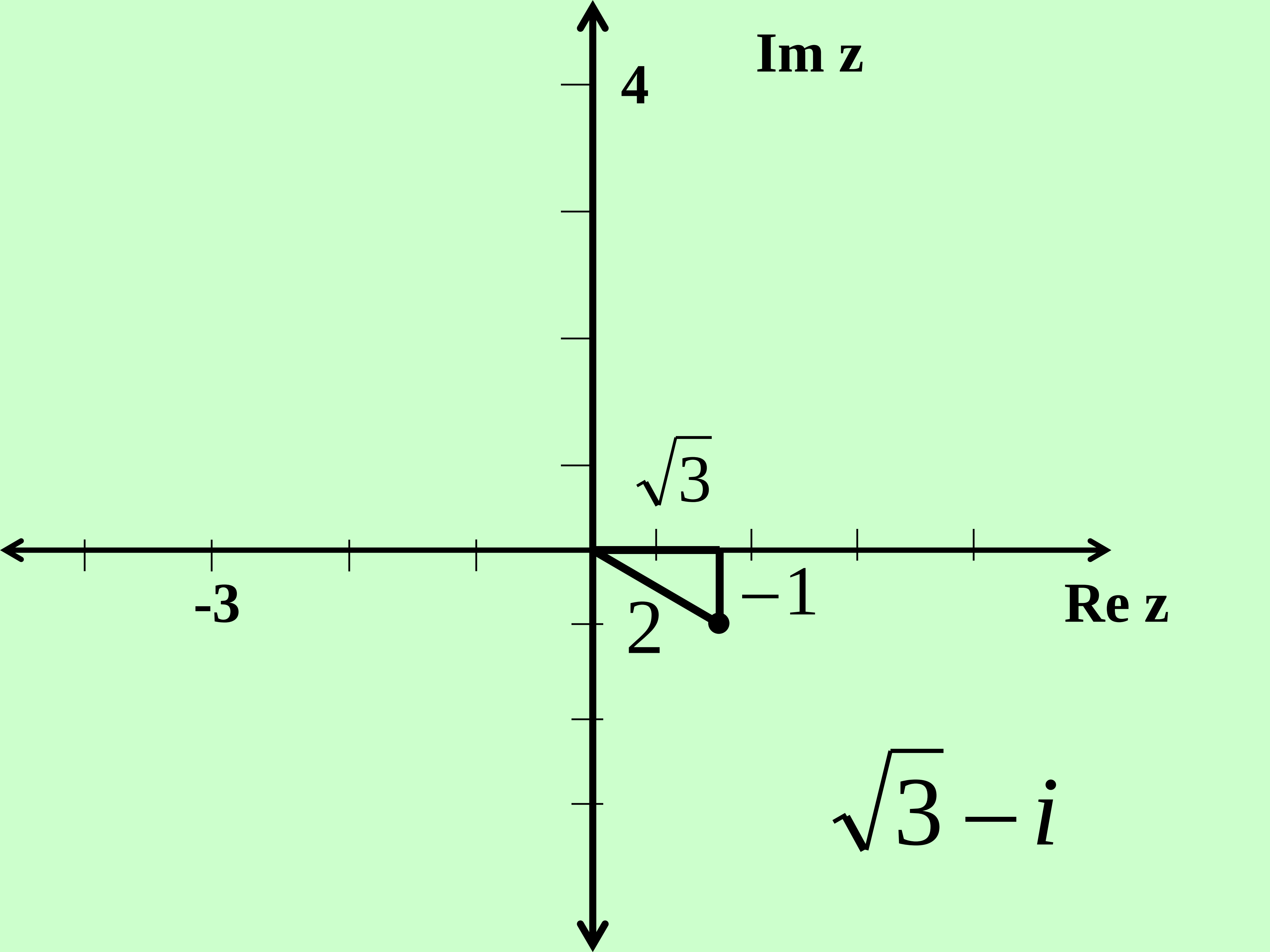
Plot the point corresponding to $z = \sqrt{3} - i$ in the complex plane, and write an expression for z in polar form.



r-ді табамыз:

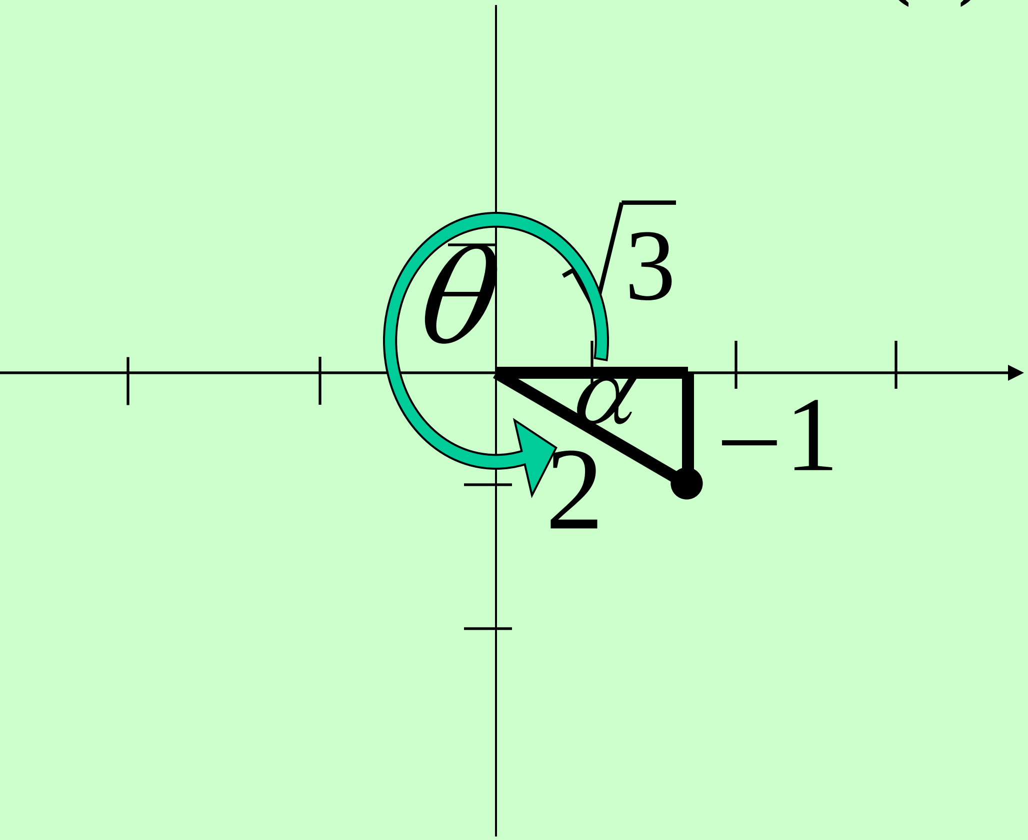
$$r = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2}$$

$$r = 2$$



Im z
4
$\sqrt{3}$
-3
-1
2
Re z
$\sqrt{3} - i$

(α) бұрышты табамыз

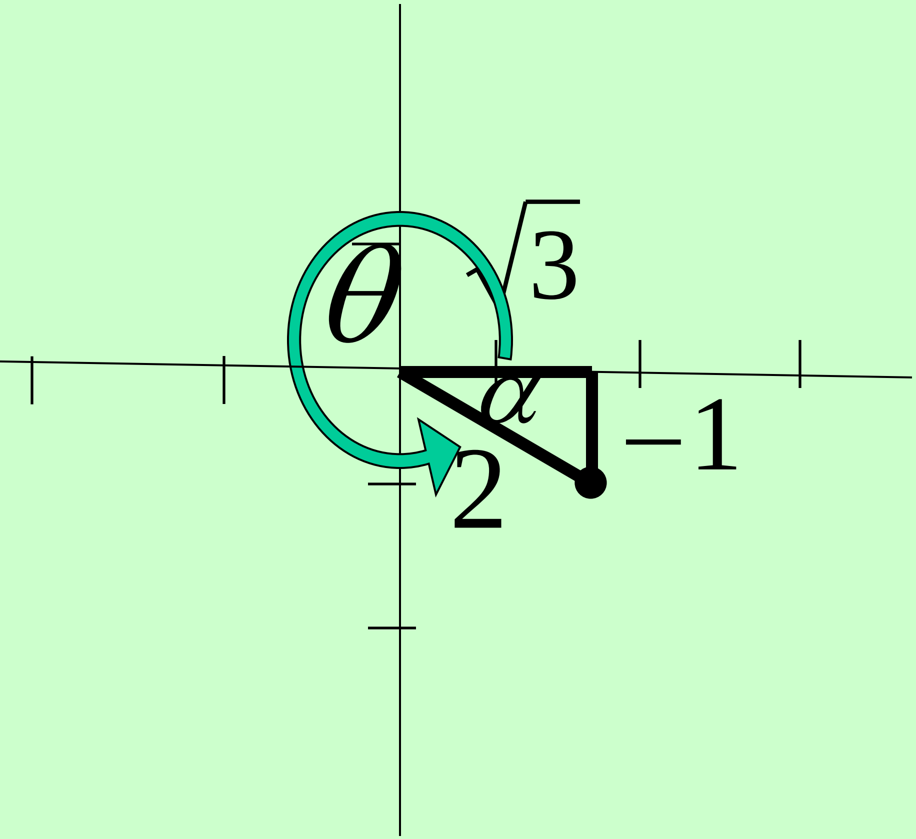


$$\tan\alpha = \left|\frac{y}{x}\right|$$

$$tan\,\alpha = \left|\frac{-1}{\sqrt{3}}\right|$$

$$\alpha = tan^{-1}\left|\frac{-1}{\sqrt{3}}\right|$$

$$\alpha = 30°$$



$$\theta = 360° - 30° = 330°$$

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 2(\cos 330° + i\sin 330°)$$

Комплекс санды алгебралық формада жазыңдар

$$2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$$

$$\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}$$

$$\sin\frac{5\pi}{6}=\frac{1}{2}$$

$$2\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$

$$=-\sqrt{3}+i$$

Комплекс сандарды Арган диаграммасында бейнелеп, тригонометриялық формада жазыңдар:

- z = 4 – 4i

$2+\sqrt{3}i$

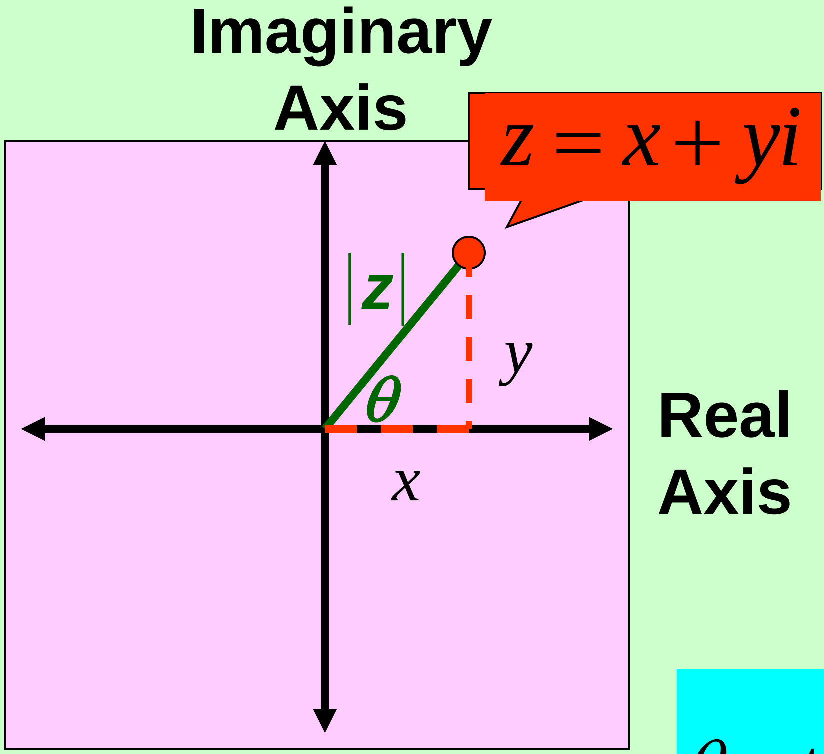
Комплекс санды Арган диаграммасында бейнелеп, алгебралық формада жазыңдар:

$$z = 2(\cos 30^o + i\sin 30^o)$$

Remember a complex number has a real part and an imaginary part. These are used to plot complex numbers on a complex plane.

$$z = x + yi$$

$$|z| = \sqrt{x^2 + y^2}$$



The magnitude or modulus of z denoted by $|z|$ is the distance from the origin to the point (x, y).

The angle formed from the real axis and a line from the origin to (x, y) is called the argument of z, with requirement that $0 \le \theta < 2\pi$.

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

modified for quadrant and so that it is between 0 and 2π

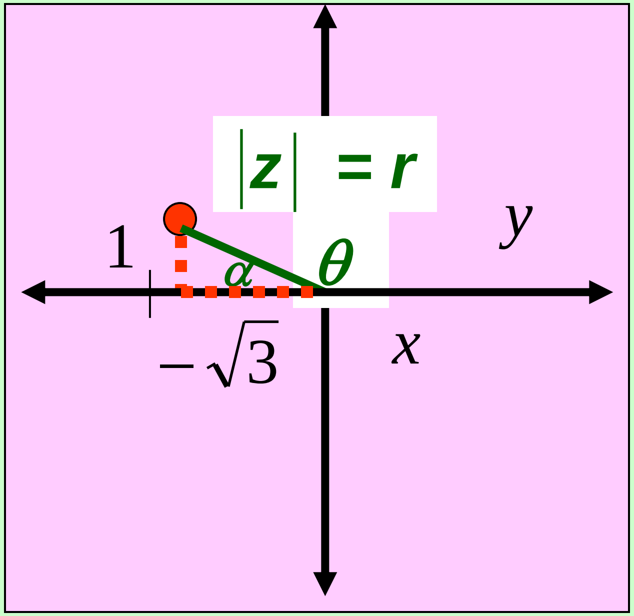
We can take complex numbers given as $z = x + yi$ and convert them to polar form. Recall the conversions:

$x = r\cos\theta$ $y = r\sin\theta$

$$z = (r\cos\theta) + (r\sin\theta)i$$

factor r out

$$= r(\cos\theta + i\sin\theta)$$



The magnitude or modulus of *z* is the same as *r.*

Plot the complex number: $z = -\sqrt{3} + i$

Find the polar form of this number.

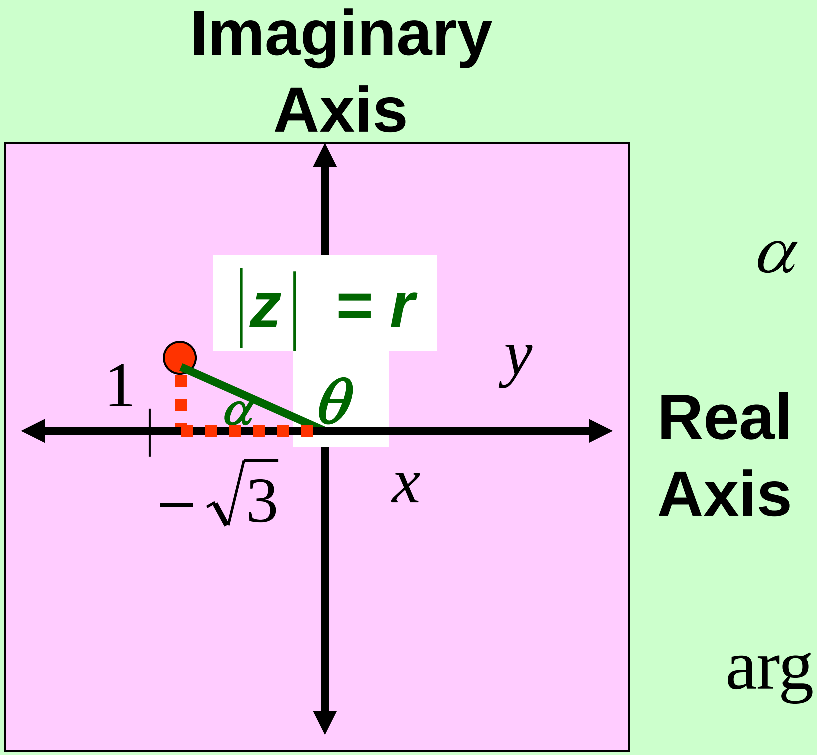
$$r = \sqrt{\left(-\sqrt{3}\right)^2 + (1)^2} = \sqrt{4} = 2$$

$$\alpha = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) \text{ but in Quad II}$$

$$\theta = \frac{5\pi}{6}$$

$$z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

The Principal Argument is between -π and π



$$\alpha = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) \text{ but in Quad II}$$

$$\theta = \frac{5\pi}{6}$$

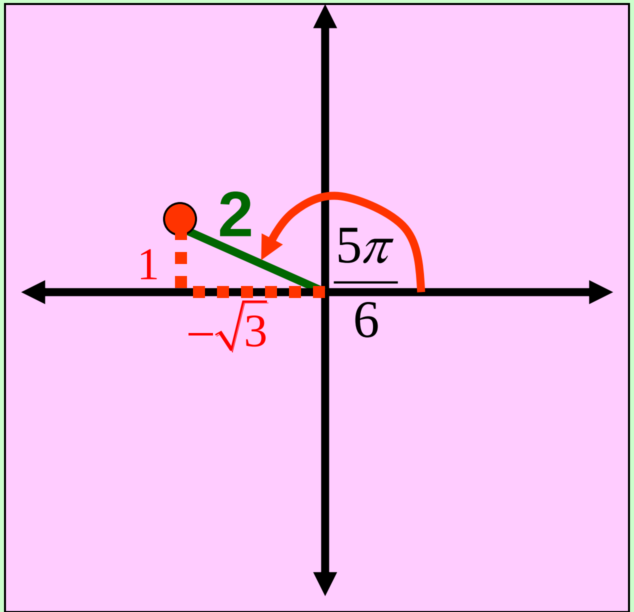
$$\arg z = \frac{5\pi}{6} \Rightarrow \textit{principal} \quad \arg = \frac{5\pi}{6}$$

$$z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

It is easy to convert from polar to rectangular form because you just work the trig functions and distribute the *r* through.

$$z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{3} + i$$

$$-\frac{\sqrt{3}}{2} \qquad \frac{1}{2}$$



If asked to plot the point and it is in polar form, you would plot the angle and radius.

Notice that is the same as plotting $-\sqrt{3} + i$

Let's try multiplying two complex numbers in polar form together.

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = \left[r_1\left(\cos\theta_1 + i\sin\theta_1\right)\right]\left[r_2\left(\cos\theta_2 + i\sin\theta_2\right)\right]$$

Look at where we started and where we ended up and see if you can make a statement as to what happens to the *r* 's and the θ 's when you multiply two complex numbers.

Multiply the Moduli and Add the Arguments

$$= r_1 r_2\left(\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right)$$

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers. Then

$$z_1 z_2 = r_1 r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

(This says to multiply two complex numbers in polar form, multiply the moduli and add the arguments)

If $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

(This says to divide two complex numbers in polar form, divide the moduli and subtract the arguments)

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers. Then

$$z_1 z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

$$z_1 z_2 = r_1 r_2\left[cis(\theta_1+\theta_2)\right]$$

If $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[cis(\theta_1-\theta_2)\right]$$

$$\text{If } z = 4\left(\cos 40^\circ + i\sin 40^\circ\right) \text{and } w = 6\left(\cos 120^\circ + i\sin 120^\circ\right),$$

$$\text{find : (a) } zw \qquad \text{(b) } z/w$$

$$zw = \left[4\left(\cos 40^\circ + i\sin 40^\circ\right)\right]\left[6\left(\cos 120^\circ + i\sin 120^\circ\right)\right]$$

$$= \left[(4 \times 6)\left(\cos\left(40^\circ + 120^\circ\right) + i\sin\left(40^\circ + 120^\circ\right)\right)\right]$$

multiply the moduli

add the arguments

(the *i* sine term will have same argument)

$$= 24\left(\cos 160^\circ + i\sin 160^\circ\right)$$

$$= 24\left(-0.93969 + 0.34202i\right)$$

$$= -22.55 + 8.21i$$

If you want the answer in rectangular coordinates simply compute the trig functions and multiply the 24 through.

$$\frac{z}{w}=\frac{4\left(\cos 40^\circ + i\sin 40^\circ\right)}{6\left(\cos 120^\circ + i\sin 120^\circ\right)}$$

$$=\frac{4}{6}\left[\cos\left(40^\circ - 120^\circ\right)+i\sin\left(40^\circ - 120^\circ\right)\right]$$

divide the moduli

subtract the arguments

$$=\frac{2}{3}\left[\cos\left(-80^\circ\right)+i\sin\left(-80^\circ\right)\right]$$

In polar form we want an angle between -180° and 180°

PRINCIPAL ARGUMENT

In rectangular coordinates: $=\frac{2}{3}\left(0.1736-0.9848i\right)=0.12-0.66i$

Using Trig Representation

- Recall that a complex number can be represented as $z = r\cdot(\cos\theta + i\cdot\sin\theta)$

- Then it follows that

$$z^2 = r(\cos\theta + i\cdot\sin\theta)\cdot r(\cos\theta + i\cdot\sin\theta)$$
$$= r^2\cdot(\cos 2\theta + i\cdot\sin 2\theta)$$

- What about z^3 ?

Abraham de Moivre
(1667 - 1754)

You can repeat this process raising complex numbers to powers. Abraham DeMoivre did this and proved the following theorem:

DeMoivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number,

then $$z^n = r^n(\cos n\theta + i\sin n\theta)$$

where $n \geq 1$ is a positive integer.

This says to raise a complex number to a power, raise the modulus to that power and multiply the argument by that power.

This theorem is used to raise complex numbers to powers. It would be a lot of work to find $\left(-\sqrt{3}+i\right)^4$

$$=\left(-\sqrt{3}+i\right)\left(-\sqrt{3}+i\right)\left(-\sqrt{3}+i\right)\left(-\sqrt{3}+i\right)$$

you would need to FOIL and multiply all of these together and simplify powers of *i* --- UGH!

Instead let's convert to polar form and use DeMoivre's Theorem.

$$r=\sqrt{\left(-\sqrt{3}\right)^2+1^2}=\sqrt{4}=2$$

$$\theta=\tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)$$ but in Quad II $$\theta=\frac{5\pi}{6}$$

$$\left(-\sqrt{3}+i\right)^4=\left[2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)\right]^4=2^4\left[\cos\left(4\times\frac{5\pi}{6}\right)+i\sin\left(4\times\frac{5\pi}{6}\right)\right]$$

$$=16\left[\cos\left(\frac{10\pi}{3}\right)+i\sin\left(\frac{10\pi}{3}\right)\right]=16\left(-\frac{1}{2}+\left(-\frac{\sqrt{3}}{2}\right)i\right)$$

$$=-8-8\sqrt{3}i$$

Solve the following over the set of complex numbers:

$$z^3 = 1$$

We know that if we cube root both sides we could get 1 but we know that there are 3 roots. So we want the complex cube roots of 1.

Using DeMoivre's Theorem with the power being a rational exponent (and therefore meaning a root), we can develop a method for finding complex roots. This leads to the following formula:

$$z^{1/n} = r^{1/n} \cdot \left(\cos\left(\frac{\theta + 360 \cdot k}{n} \right) + i \cdot \sin\left(\frac{\theta + 360 \cdot k}{n} \right) \right)$$

$$\text{where } k = 0, 1, 2, \ldots, n-1$$

Let's try this on our problem. We want the cube roots of 1.

We want cube root so our *n* = 3. Can you convert 1 to polar form? (hint: 1 = 1 + 0*i*)

$$r=\sqrt{(1)^2+(0)^2}=1 \qquad \theta=\tan^{-1}\left(\frac{0}{1}\right)=0$$

$$z_k=\sqrt[3]{1}\left[\cos\left(\frac{0}{3}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{0}{3}+\frac{2k\pi}{3}\right)\right],\ \text{for } k=0,1,2$$

We want cube root so use 3 numbers here

Once we build the formula, we use it first with *k* = 0 and get one root, then with *k* = 1 to get the second root and finally with *k* = 2 for last root.

$$z^{1/n}=r^{1/n}\cdot\left(\cos\left(\frac{\theta+360\cdot k}{n}\right)+i\cdot\sin\left(\frac{\theta+360\cdot k}{n}\right)\right)$$

$$z_k = \sqrt[3]{1}\left[\cos\left(\frac{0}{3}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{0}{3}+\frac{2k\pi}{3}\right)\right], \text{ for } k = 0, 1, 2$$

$$z_0 = \sqrt[3]{1}\left[\cos\left(\frac{0}{3}+\frac{2(0)\pi}{3}\right)+i\sin\left(\frac{0}{3}+\frac{2(0)\pi}{3}\right)\right] = 1\left[\cos(0)+i\sin(0)\right] = \boxed{1}$$

Here's the root we already knew.

$$z_1 = \sqrt[3]{1}\left[\cos\left(\frac{0}{3}+\frac{2(1)\pi}{3}\right)+i\sin\left(\frac{0}{3}+\frac{2(1)\pi}{3}\right)\right]$$

$$= 1\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right] = \boxed{-\frac{1}{2}+\frac{\sqrt{3}}{2}i}$$

$$z_2 = \sqrt[3]{1}\left[\cos\left(\frac{0}{3}+\frac{2(2)\pi}{3}\right)+i\sin\left(\frac{0}{3}+\frac{2(2)\pi}{3}\right)\right]$$

$$= 1\left[\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right] = \boxed{-\frac{1}{2}-\frac{\sqrt{3}}{2}i}$$

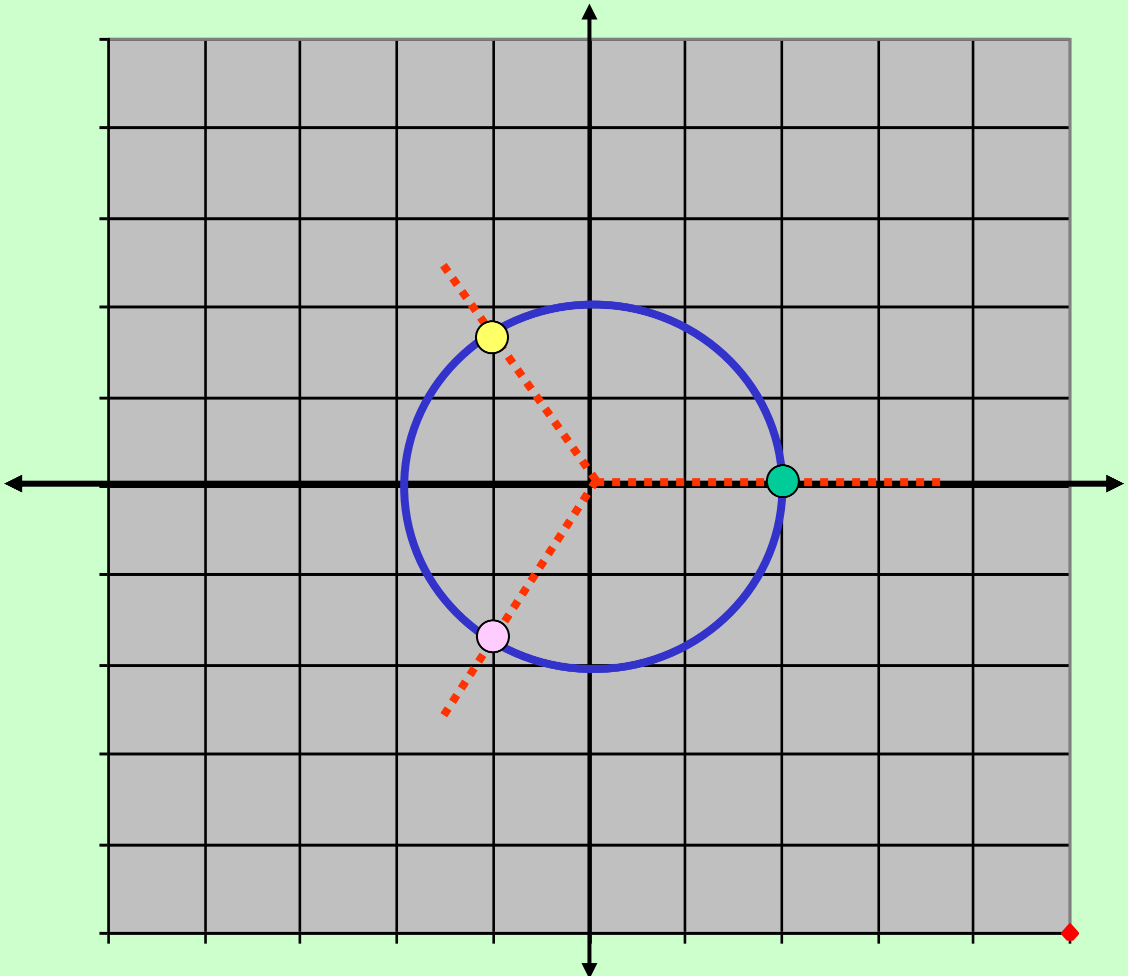
If you cube any of these numbers you get 1. (Try it and see!)

We found the cube roots of 1 were: $1, \quad -\frac{1}{2}+\frac{\sqrt{3}}{2}i, \quad -\frac{1}{2}-\frac{\sqrt{3}}{2}i$

about 0.9

Let's plot these on the complex plane

each line is 1/2 unit



Notice each of the complex roots has the same magnitude (1). Also the three points are evenly spaced on a circle. This will always be true of complex roots.

DeMoivre's Theorem

- In general $(a + b\boldsymbol{i})^n$ is

$$z^n = r^n \cdot \left(\cos\left(n\cdot\theta\right) + i\cdot\sin\left(n\cdot\theta\right)\right)$$

Where $n \geq 1$

- Apply to

$$\left(3\left(\cos 330^\circ + i\cdot\sin 330^\circ\right)\right)^4$$

- Try

$$\left(-\frac{\sqrt{2}}{2} + i\cdot\frac{\sqrt{2}}{2}\right)^{12}$$

Using De Moivre to Find Roots

- Note that there will be *n* such roots

$$z^{1/n} = r^{1/n} \cdot \left(\cos\left(\frac{\theta + 360 \cdot k}{n} \right) + i \cdot \sin\left(\frac{\theta + 360 \cdot k}{n} \right) \right)$$

  – One each for k = 0, k = 1, … k = n – 1

- Find the two square roots of $-1 + i \cdot \sqrt{3}$
  – Represent as z = r cis θ
  – What is r?
  – What is θ?