1. Given that $z = \cos\theta + \text{i} \sin \theta$ show that

(a) $\text{Im}\left(z^n + \frac{1}{z^n}\right) = 0, n \in \mathbb{Z}^+$;

(2)

(b) $\text{Re}\left(\frac{z-1}{z+1}\right) = 0, z \neq -1.$

(5)
(Total 7 marks)

2. Consider the polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$.

Given that $1 + \text{i}$ and $1 - 2\text{i}$ are zeros of $p(x)$, find the values of a, b, c and d.

(Total 7 marks)

3. Find, in its simplest form, the argument of $(\sin\theta + \text{i}\,(1 - \cos\theta))^2$ where θ is an acute angle.

(Total 7 marks)

4. $z_1 = \left(1 + \text{i}\sqrt{3}\right)^m$ and $z_2 = (1 - \text{i})^n$.

(a) Find the modulus and argument of z_1 and z_2 in terms of m and n, respectively.

(6)

(b) **Hence**, find the smallest positive integers m and n such that $z_1 = z_2$.

(8)
(Total 14 marks)

5. The complex numbers $z_1 = 2 - 2\text{i}$ and $z_2 = 1 - \text{i}\sqrt{3}$ are represented by the points A and B respectively on an Argand diagram. Given that O is the origin,

(a) find AB, giving your answer in the form $a\sqrt{b-\sqrt{3}}$, where $a, b \in \mathbb{Z}^+$;

(3)

(b) calculate $\text{A}\hat{\text{O}}\text{B}$ in terms of π.

(3)
(Total 6 marks)

6. (a) Factorize $z^3 + 1$ into a linear and quadratic factor.

(2)

Let $\gamma = \dfrac{1+\text{i}\sqrt{3}}{2}$.

(b) (i) Show that γ is one of the cube roots of -1.

(ii) Show that $\gamma^2 = \gamma - 1$.

(iii) Hence find the value of $(1 - \gamma)^6$.

(9)

The matrix $\boldsymbol{A}$ is defined by $\boldsymbol{A} = \begin{pmatrix} \gamma & 1 \\ 0 & \frac{1}{\gamma} \end{pmatrix}$.

(c) Show that $\boldsymbol{A}^2 - \boldsymbol{A} + \boldsymbol{I} = \boldsymbol{0}$, where $\boldsymbol{0}$ is the zero matrix.

(4)

(d) Deduce that

(i) $\boldsymbol{A}^3 = -\boldsymbol{I}$;

(ii) $\boldsymbol{A}^{-1} = \boldsymbol{I} - \boldsymbol{A}$.

(5)
(Total 20 marks)

7. Consider $\omega = \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)$.

(a) Show that

(i) $\omega^3 = 1$;

(ii) $1 + \omega + \omega^2 = 0$.

(5)

(b) (i) Deduce that $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{4\pi}{3}\right)} = 0$.

(ii) Illustrate this result for $\theta = \frac{\pi}{2}$ on an Argand diagram.

(4)

(c) (i) Expand and simplify $F(z) = (z - 1)(z - \omega)(z - \omega^2)$ where z is a complex number.

(ii) Solve $F(z) = 7$, giving your answers in terms of ω.

(7)
(Total 16 marks)

8. Consider the complex number $\omega = \frac{z + \mathrm{i}}{z + 2}$, where $z = x + \mathrm{i}y$ and $\mathrm{i} = \sqrt{-1}$.

(a) If $\omega = \mathrm{i}$, determine z in the form $z = r \operatorname{cis} \theta$.

(6)

(b) Prove that $\omega = \dfrac{(x^2 + 2x + y^2 + y) + \mathrm{i}(x + 2y + 2)}{(x + 2)^2 + y^2}$.

(3)

(c) **Hence** show that when Re(ω) = 1 the points (x, y) lie on a straight line, l_1, and write down its gradient.

(4)

(d) Given arg (z) = arg(ω) = $\dfrac{\pi}{4}$, find $|z|$.

(6)
(Total 19 marks)

9. Find the three cube roots of the complex number 8i. Give your answers in the form $x + \mathrm{i}y$.

(Total 8 marks)

10. (a) Solve the equation $z^3 = -2 + 2\mathrm{i}$, giving your answers in modulus–argument form.

(6)

(b) **Hence** show that one of the solutions is 1 + i when written in Cartesian form.

(1)
(Total 7 marks)

11. Let $w = \cos\dfrac{2\pi}{5} + \mathrm{i}\sin\dfrac{2\pi}{5}$.

(a) Show that w is a root of the equation $z^5 - 1 = 0$.

(3)

(b) Show that $(w - 1)(w^4 + w^3 + w^2 + w + 1) = w^5 - 1$ and deduce that $w^4 + w^3 + w^2 + w + 1 = 0$.

(3)

(c) **Hence** show that $\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} = -\dfrac{1}{2}$.

(6)
(Total 12 marks)

12. Consider the complex numbers $z = 1 + 2\text{i}$ and $w = 2 + a\text{i}$, where $a \in \mathbb{R}$.

Find a when

(a) $|w| = 2|z|$;

(3)

(b) $\text{Re}(zw) = 2\,\text{Im}(zw)$.

(3)
(Total 6 marks)